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2005 J. Phys. A: Math. Gen. 38 10971

(http://iopscience.iop.org/0305-4470/38/50/007)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 10971–10987 doi:10.1088/0305-4470/38/50/007

Two-party Bell inequalities derived from combinatorics via triangular elimination

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Received 8 April 2005, in final form 20 September 2005
Published 30 November 2005
Online at stacks.iop.org/JPhysA/38/10971

Abstract

We establish a relation between the two-party Bell inequalities for two-valued measurements and a high-dimensional convex polytope called the cut polytope in polyhedral combinatorics. Using this relation, we propose a method, *triangular elimination*, to derive tight Bell inequalities from facets of the cut polytope. This method gives two hundred million inequivalent tight Bell inequalities from currently known results on the cut polytope. In addition, this method gives general formulae which represent families of infinitely many Bell inequalities. These results can be used to examine general properties of Bell inequalities.

PACS numbers: 03.65.Ud, 02.10.Ud

1. Introduction

Bell inequalities have been intensively studied in quantum theory [1, 2], and it is known that they can be obtained from the structure of certain convex polytopes [3–5]. Bell inequalities are not the only example of the use of convex polytopes in quantum theory. In a pioneering paper, McRae and Davidson [6] used the theory of convex polytopes to obtain inequalities bounding the range of possible solutions to some problems in quantum mechanics. Their method is summarized as follows: first they prove that the possible solutions form a convex polytope and obtain the set of vertices of the polytope. Then they obtain a minimum set of inequalities that describe the polytope using a convex hull algorithm. Interestingly, one of the polytopes that McRae and Davidson considered coincides with the *correlation polytope* Pitowsky introduced in [3], in connection with Bell inequalities. This polytope arises in many fields under different names, and a comprehensive source for results on this polytope and the related cut polytope (described later) is the book by Deza and Laurent [7].

In this paper, we consider the results of correlation experiments between two parties, where one party has m_A choices of possible two-valued measurements and the other party has m_B choices. The relevant polytope can be described as follows. The results of a series of such correlation experiments are represented as a vector of $m_A + m_B + m_A m_B$ probabilities. In classical mechanics, the set of vectors which are possible results of a correlation experiment forms an $(m_A + m_B + m_A m_B)$-dimensional convex polytope which is a projection of the correlation polytope onto the complete bipartite graph K_{m_A,m_B}. A Bell inequality is nothing but a linear inequality satisfied by all the points in such a polytope. A tight Bell inequality is a Bell inequality which cannot be represented as a positive weighted sum of other Bell inequalities and defines a facet of the polytope. Two examples of these facet-defining inequalities are the non-negativity inequality and the Clauser–Horne–Shimony–Holt (CHSH) inequality [8].

By considering these polytopes, Bell's original inequality [9], the CHSH and many other known Bell inequalities can be understood in a unified manner. Fine's necessary and sufficient conditions [10] for $m_A = m_B = 2$ can be seen as the complete inequality representation of the correlation polytope of the complete bipartite graph $K_{2,2}$. Pitowsky and Svozil [11] and Collins and Gisin [12] apply convex hull algorithms to obtain a complete list of tight Bell inequalities in other experimental settings. As a result, we know the complete list of Bell inequalities in the cases $m_A = 2$ [12], $(m_A, m_B) = (3, 3)$ [11] and $(m_A, m_B) = (3, 4)$ [12]. Several software packages for convex hull computation such as cdd [13] and lrs [14] are readily available. It is unlikely, however, that there exists a compact representation of the complete set of Bell inequalities in arbitrarily large settings. This follows from the fact that testing whether a vector of correlations lies in the correlation polytope of the bipartite graph K_{m_A,m_B} is NP-complete [15]. Therefore, it is natural to look for families of Bell inequalities, especially those that are facet producing. In this direction, Collins and Gisin [12] give a family I_{mm22} of Bell inequalities in the case $m_A = m_B = m$ for general m. In addition there are several extensions [12, 16, 17] of the CHSH inequality for multi-valued measurements.

In the field of polyhedral combinatorics, a polytope isomorphic to the correlation polytope, called the *cut polytope*, has been studied in great detail [7]. The correlation and cut polytopes are isomorphic via a linear mapping [18] and so the inequalities representing them correspond one-to-one. This relationship enables us to apply results for the cut polytope to the study of Bell inequalities. Related to this, Pironio [19] uses lifting, which is a common approach in combinatorial optimization, to generate tight Bell inequalities for a larger system from those for a smaller system. Since the mathematical description of the facet structure of cut polytopes is simpler than that for correlation polytopes, the former are preferred in polyhedral combinatorics. Large classes of facets for the cut polytope $\mathrm{CUT}_n^\square$ of the complete graph K_n are known for general n [7], and a complete or conjectured complete list of all facets for $\mathrm{CUT}_n^\square$ is known for $n \leqslant 9$ [20]. We make use of these results in this paper.

The cut polytope of the complete graph has been the most extensively studied. However, the case we are interested in corresponds to the correlation polytope of the complete bipartite graph K_{m_A,m_B}, which maps to the cut polytope of the complete tripartite graph K_{1,m_A,m_B}. To overcome this gap, we introduce a method called triangular elimination to convert an inequality valid for $\mathrm{CUT}_n^\square$ to another inequality valid for $\mathrm{CUT}^\square(K_{1,m_A,m_B})$, which is then converted to a Bell inequality via the isomorphism. The CHSH inequality and some of the other previously known inequalities can be explained in this manner. More importantly, triangular elimination converts a facet inequality of $\mathrm{CUT}_n^\square$ to a facet inequality of $\mathrm{CUT}^\square(K_{1,m_A,m_B})$, which corresponds to a tight Bell inequality.

A complete list of facets of $\mathrm{CUT}_n^\square$ for $n \leqslant 7$ and a conjectured complete list for $n = 8, 9$ are known. We apply triangular elimination to these facets to obtain 201 374 783 tight Bell inequalities. On the other hand, several formulae which represent many different inequalities valid for $\mathrm{CUT}_n^\square$ are known. We apply triangular elimination to these formulae to obtain new families of Bell inequalities. We discuss their properties such as tightness and inclusion of the CHSH inequality.

The rest of this paper is organized as follows. In section 2, we introduce triangular elimination to derive tight Bell inequalities from facets of the cut polytope of the complete graph, and show its properties. We also give a computational result on the number of Bell inequalities obtained by triangular elimination. In section 3, we apply triangular elimination to some of the known classes of facets of the cut polytope of the complete graph to obtain general formulae representing many Bell inequalities. Section 4 concludes the paper by giving the relation of our result to some of the open problems posed in [2].

2. Triangular elimination

2.1. Bell inequalities and facets of cut polytopes

Consider a system composed of subsystems A (Alice) and B (Bob). Suppose that on both subsystems, one of m_A observables for Alice and one of m_B observables for Bob are measured. For each observable, the outcome is one of the two values (in the rest of the paper, we label the outcomes as 0 or 1). The experiment is repeated a large number of times. The result of such a correlation experiment consists of the probability distribution of the $m_\mathrm{A}m_\mathrm{B}$ joint measurements by both parties. Throughout this paper, we represent the experimental result as a vector $\boldsymbol{q}$ in $(m_\mathrm{A} + m_\mathrm{B} + m_\mathrm{A}m_\mathrm{B})$-dimensional space in the following manner: q_{A_i}, q_{B_j} and $q_{A_iB_j}$ correspond to the probabilities $\Pr[A_i = 1]$, $\Pr[B_j = 1]$ and $\Pr[A_i = 1 \wedge B_j = 1]$, respectively.

In classical mechanics, the result of a correlation experiment must correspond to a probability distribution over all *classical configurations*, where a classical configuration is an assignment of the outcomes $\{0, 1\}$ to each of the $m_\mathrm{A} + m_\mathrm{B}$ observables. The experimental result has a *local hidden variable model* if and only if a given experimental result can be interpreted as a result of such a classical correlation experiment.

Bell inequalities are valid linear inequalities for every experimental result which has a local hidden variable model. Specifically using the above formulation, we represent a Bell inequality in the form

$$\sum_{1\leqslant i\leqslant m_\mathrm{A}} b_{A_i}q_{A_i} + \sum_{1\leqslant j\leqslant m_\mathrm{B}} b_{B_j}q_{B_j} + \sum_{1\leqslant i\leqslant m_\mathrm{A},1\leqslant j\leqslant m_\mathrm{B}} b_{A_iB_j}q_{A_iB_j} \leqslant b_0,$$

for suitably chosen constants b_x.

For example, Clauser, Horn, Shimony and Holt [8] have shown that the following CHSH inequality is a valid Bell inequality:

$$-q_{A_1} - q_{B_1} + q_{A_1B_1} + q_{A_1B_2} + q_{A_2B_1} - q_{A_2B_2} \leqslant 0.$$

In general, the set of all experimental results with a local hidden variable model forms a convex polytope with extreme points corresponding to the classical configurations. If the results of the experiment are in the above form, the polytope is called a *correlation polytope*, a name introduced by Pitowsky [21]. (Such polyhedra have been discovered and rediscovered several times, see for instance Deza and Laurent [7].) From such a viewpoint, Bell inequalities can be considered as the boundary, or face inequalities, of that polytope. Since every polytope is the intersection of finitely many half spaces represented by linear inequalities, every

Bell inequality can be represented by a convex combination of finitely many extremal inequalities. Such extremal inequalities are called *tight* Bell inequalities. Non-extremal inequalities are called *redundant*.

In polytopal theory, the maximal extremal faces of a polytope are called *facets*. Therefore, tight Bell inequalities are facet inequalities of the polytope formed by experimental results with a local hidden variable model. Note that for a given linear inequality $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{q} \leqslant b_0$ and d-dimensional polytope, the face represented by the inequality is a facet of that polytope if and only if the dimension of the convex hull of the extreme points for which the equality holds is $d-1$.

2.1.1. Cut polytope of complete tripartite graph. We introduce a simple representation of an experimental setting as a graph. Consider a graph which consists of vertices corresponding to observables A_i or B_j and edges corresponding to joint measurements between A_i and B_j. In addition, to represent probabilities which are the results of single (not joint) measurements, we introduce a vertex X (which represents the trace out operation of the other party) and edges between X and A_i for every $1 \leqslant i \leqslant m_{\mathrm{A}}$, and between X and B_j for every $1 \leqslant j \leqslant m_{\mathrm{B}}$. This graph is a complete tripartite graph since there exist edges between each party of vertices (observables) $\{X\}$, $\{A_i\}$ and $\{B_j\}$. Using this graph, we can conveniently represent either the result probabilities or the coefficients of a Bell inequality as edge labels. We denote this graph by $K_{1,m_{\mathrm{A}},m_{\mathrm{B}}}$.

In polyhedral combinatorics, a polytope affinely isomorphic to the correlation polytope has been well studied. Specifically, if we consider the probabilities $x_{A_iB_j} = \Pr[A_i \neq B_j]$ instead of $q_{A_iB_j} = \Pr[A_i = 1 \wedge B_j = 1]$ for each edge, the probabilities form a polytope called the *cut polytope*. Thus, the cut polytope is another formulation of the polytope formed by Bell inequalities.

A *cut* in a graph is an assignment of $\{0, 1\}$ to each vertex, 1 to an edge between vertices with different values assigned, and 0 to an edge between vertices with the same values assigned. In the above formulation, each cut corresponds to a classical configuration. Note that since the 0, 1 exchange of all values of vertices yields the same edge cut, we can without loss of generality assume that the vertex X is always assigned the label 0.

Let the *cut vector* $\boldsymbol{\delta}'(S') \in \mathbb{R}^{\{XA_i\}\cup\{XB_j\}\cup\{A_iB_j\}}$ for some cut S' be $\delta'_{uv}(S') = 1$ if vertices u and v are assigned different values, and 0 if assigned the same values. Then, the convex combination of all the cut vectors $\mathrm{CUT}^{\square}\big(K_{1,m_{\mathrm{A}},m_{\mathrm{B}}}\big) = \big\{\boldsymbol{x} = \sum_{S':\mathrm{cut}} \lambda_{S'}\boldsymbol{\delta}'(S') \,\big|\, \sum_{S':\mathrm{cut}} \lambda_{S'} = 1 \text{ and } \lambda_{S'} \geqslant 0\big\}$ is called the cut polytope of the complete tripartite graph. The cut polytope has full dimension. Therefore, $\dim\big(\mathrm{CUT}^{\square}\big(K_{1,m_{\mathrm{A}},m_{\mathrm{B}}}\big)\big) = m_{\mathrm{A}} + m_{\mathrm{B}} + m_{\mathrm{A}}m_{\mathrm{B}}$.

In this formulation, a tight Bell inequality $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{q} \leqslant b_0$ corresponds to a facet inequality $\boldsymbol{a}'^{\mathrm{T}}\boldsymbol{x} \leqslant a_0$ of the cut polytope. The affine isomorphisms between them are

$$\begin{cases} x_{XA_i} = q_{A_i}, \\ x_{XB_j} = q_{B_j}, \\ x_{A_iB_j} = q_{A_i} + q_{B_j} - 2q_{A_iB_j}, \end{cases} \qquad \begin{cases} q_{A_i} = x_{XA_i}, \\ q_{B_j} = x_{XB_j}, \\ q_{A_iB_j} = \frac{1}{2}\big(x_{XA_i} + x_{XB_j} - x_{A_iB_j}\big). \end{cases} \tag{1}$$

Actually, because cut polytopes are symmetric under the switching operation (explained in section 2.4) we can assume that the right-hand side of a facet inequality of the cut polytope is always 0. This means that a given Bell inequality is tight if and only if for the corresponding facet inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leqslant 0$ of the cut polytope, there exist $m_{\mathrm{A}} + m_{\mathrm{B}} + m_{\mathrm{A}}m_{\mathrm{B}} - 1$ linearly independent cut vectors $\boldsymbol{\delta}'(S')$ for which $\boldsymbol{a}'^{\mathrm{T}}\boldsymbol{\delta}'(S') = 0$.

For example, there exists a facet inequality $-x_{A_1B_1} - x_{A_1B_2} - x_{A_2B_1} + x_{A_2B_2} \leqslant 0$ for $\mathrm{CUT}^{\square}\big(K_{1,m_{\mathrm{A}},m_{\mathrm{B}}}\big)$, $1 \leqslant m_{\mathrm{A}}, m_{\mathrm{B}}$, which corresponds to the CHSH inequality. Therefore, the CHSH inequality is tight in addition to being valid.

A consequence of the above affine isomorphisms is that any theorem concerning facets of the cut polytope can be immediately translated to give a corresponding theorem for tight Bell inequalities. Recently, Collins and Gisin [12] gave the following conjecture about the tightness of Bell inequalities: if a Bell inequality $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{q} \leqslant b_0$ is tight in a given setting $m_{\mathrm{A}}, m_{\mathrm{B}}$, then for each $m'_{\mathrm{A}} \geqslant m_{\mathrm{A}}$ and $m'_{\mathrm{B}} \geqslant m_{\mathrm{B}}$, the inequality $\boldsymbol{b}'^{\mathrm{T}}\boldsymbol{q}' \leqslant b_0$ is also tight. Here $\boldsymbol{b}'$ is the vector $b'_{uv} = b_{uv}$ if the vertices (observables) u, v appear in $\boldsymbol{b}$ and is zero otherwise. They gave empirical evidence for this conjecture based on numerical experiments. In fact, a special case of the *zero-lifting theorem* by De Simone [22] gives a proof of their conjecture.

2.2. Triangular elimination

2.2.1. Cut polytope of complete graph. In the previous section we saw that the problem of enumerating tight Bell inequalities is equivalent to that of enumerating facet inequalities of the cut polytope of a corresponding complete tripartite graph. The properties of facet inequalities of the cut polytope of the complete graph K_n are well studied and there are rich results. For example, several general classes of facet inequalities with relatively simple representations are known.

For $n \leqslant 7$ the complete list of facets is known [23], and for $n = 8, 9$ a conjectured complete list is known [20, 24]. In addition, the symmetry of the polytope is also well understood. We show how to apply such results to our complete tripartite graph case.

First, we introduce the cut polytope of complete graph. The graph is denoted by K_n, has n vertices and has an edge between each pair of vertices. As before, a cut is an assignment of $\{0, 1\}$ to each vertex, and an edge is labelled by 1 if the endpoints of the edge are labelled differently or 0 if labelled the same. The cut vectors $\delta(S)$ of the complete graph are defined in the same manner as before. The set of all convex combinations of cut vectors $\mathrm{CUT}^{\square}(K_n) = \left\{\boldsymbol{x} = \sum_{S:\mathrm{cut}} \lambda_S \boldsymbol{\delta}(S) \,\middle|\, \sum_{S:\mathrm{cut}} \lambda_S = 1 \text{ and } \lambda_S \geqslant 0\right\}$ is called the cut polytope of the complete graph. $\mathrm{CUT}^{\square}(K_n)$ is also written as $\mathrm{CUT}^{\square}_n$.

In contrast to the complete tripartite graph, the space on which the cut polytope of the complete graph exists has elements corresponding to probabilities of joint measurement by the same party. Because of the restrictions of quantum mechanics, such joint measurements are prohibited. Therefore, if we want to generate tight Bell inequalities from the known facet inequalities of the cut polytope of the complete graph, we must transform the inequalities to eliminate joint measurement terms. In polyhedral terms, $\mathrm{CUT}^{\square}\left(K_{1,m_{\mathrm{A}},m_{\mathrm{B}}}\right)$ is a projection of $\mathrm{CUT}^{\square}(K_n)$ onto a lower dimensional space.

2.2.2. Definition of triangular elimination. A well-known method for projecting a polytope is called Fourier–Motzkin elimination. This is essentially the summation of two facet inequalities to cancel out the target term. For example, it is well known that the triangle inequality $x_{uv} - x_{uw} - x_{wv} \leqslant 0$, for any three vertices u, v and w, is valid for the cut polytope of the complete graph. In fact, Bell's original inequality [9] is essentially this inequality. The CHSH inequality $-x_{A_1B_1} - x_{A_1B_2} - x_{A_2B_1} + x_{A_2B_2} \leqslant 0$ is the sum of $x_{A_1A_2} - x_{A_1B_1} - x_{A_2B_1} \leqslant 0$ and $x_{A_2B_2} - x_{A_1B_2} - x_{A_1A_2} \leqslant 0$ (see figure 1).

In general, the result of Fourier–Motzkin elimination is not necessarily a facet. For example, it is known that the pentagonal inequality

$$x_{XA_1} + x_{XA_2} - x_{XB_1} - x_{XB_2} + x_{A_1A_2} - x_{A_1B_1} - x_{A_1B_2} - x_{A_2B_1} - x_{A_2B_2} + x_{B_1B_2} \leqslant 0 \qquad (2)$$

is a facet inequality of $\mathrm{CUT}^{\square}(K_5)$. If we eliminate joint measurement terms $x_{A_1A_2}$ and $x_{B_1B_2}$ by adding triangle inequalities $x_{A_1B_2} - x_{A_1A_2} - x_{A_2B_2} \leqslant 0$ and $x_{A_2B_1} - x_{B_1B_2} - x_{A_2B_2} \leqslant 0$, the result is $x_{XA_1} + x_{XA_2} - x_{XB_1} - x_{XB_2} - x_{A_1B_1} - 3x_{A_2B_2} \leqslant 0$. Therefore, this inequality

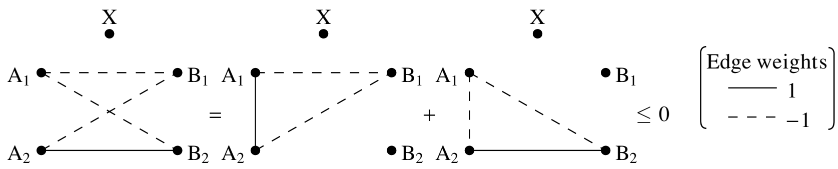

Figure 1. The most simple case of triangular elimination. The sum of two triangle inequalities is the CHSH inequality.

is a valid inequality for $\mathrm{CUT}^{\square}(K_{1,3,3})$. However, the inequality is a summation of four valid triangle inequalities for $\mathrm{CUT}^{\square}(K_{1,3,3})$, namely, $x_{XA_1} - x_{XB_1} - x_{A_1B_1} \leqslant 0$, $x_{XA_2} - x_{XB_2} - x_{A_2B_2} \leqslant 0$, $x_{XA_2} - x_{XB_2} - x_{A_2B_2} \leqslant 0$ and $-x_{XA_2} + x_{XB_2} - x_{A_2B_2} \leqslant 0$. This means that the inequality with eliminated terms is redundant.

Fourier–Motzkin elimination often produces large numbers of redundant inequalities, causing the algorithm to be computationally intractable when iterated many times. Therefore, it is important to find situations where the new inequalities found are guaranteed to be tight.

The difference between the two examples is that in the CHSH case, the second triangle inequality introduced a new vertex B_2 where 'new' means that the first triangle inequality had no term with subscript labelled B_2. Generalizing this operation, we will show that Fourier–Motzkin elimination by triangle inequalities, which introduce new vertices, is almost always guaranteed to yield non-redundant inequalities. We call the operation *triangular elimination*.

Definition 2.1 (triangular elimination)**.** *For a given valid inequality for* $\mathrm{CUT}^{\square}\left(K_{1+n_\mathrm{A}+n_\mathrm{B}}\right)$

$$\sum_{1\leqslant i\leqslant n_\mathrm{A}} a_{XA_i}x_{XA_i} + \sum_{1\leqslant j\leqslant n_\mathrm{B}} a_{XB_j}x_{XB_j} + \sum_{1\leqslant i\leqslant n_\mathrm{A},1\leqslant j\leqslant n_\mathrm{B}} a_{A_iB_j}x_{A_iB_j} + \sum_{1\leqslant i<i'\leqslant n_\mathrm{A}} a_{A_iA_{i'}}x_{A_iA_{i'}} + \sum_{1\leqslant j<j'\leqslant n_\mathrm{B}} a_{B_jB_{j'}}x_{B_jB_{j'}} \leqslant a_0, \tag{3}$$

the triangular elimination is defined as follows:

$$\sum_{1\leqslant i\leqslant n_\mathrm{A}} a_{XA_i}x_{XA_i} + \sum_{1\leqslant j\leqslant n_\mathrm{B}} a_{XB_j}x_{XB_j} + \sum_{1\leqslant i\leqslant n_\mathrm{A},1\leqslant j\leqslant n_\mathrm{B}} a_{A_iB_j}x_{A_iB_j} + \sum_{1\leqslant i<i'\leqslant n_\mathrm{A}} \left(a_{A_iA_{i'}}x_{A_iB'_{A_iA_{i'}}} - \left|a_{A_iA_{i'}}\right|x_{A_{i'}B'_{A_iA_{i'}}}\right) + \sum_{1\leqslant j<j'\leqslant n_\mathrm{B}} \left(a_{B_jB_{j'}}x_{A'_{B_jB_{j'}}B_j} - \left|a_{A_jA_{j'}}\right|x_{A'_{B_jB_{j'}}B_{j'}}\right) \leqslant a_0. \tag{4}$$

This is an inequality for $\mathrm{CUT}^{\square}\left(K_{1,m_\mathrm{A},m_\mathrm{B}}\right)$*, where* $m_\mathrm{A} = n_\mathrm{A} + \frac{n_\mathrm{B}(n_\mathrm{B}-1)}{2}$ *and* $m_\mathrm{B} = n_\mathrm{B} + \frac{n_\mathrm{A}(n_\mathrm{A}-1)}{2}$*. We denote* (3) *by* $\boldsymbol{a}^\mathrm{T}\boldsymbol{x} \leqslant 0$, $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^{\frac{(n_\mathrm{A}+n_\mathrm{B})(n_\mathrm{A}+n_\mathrm{B}+1)}{2}}$*, and* (4) *by* $\boldsymbol{a}'^\mathrm{T}\boldsymbol{x}' \leqslant 0$, $\boldsymbol{a}', \boldsymbol{x}' \in \mathbb{R}^{m_\mathrm{A}+m_\mathrm{B}+m_\mathrm{A}m_\mathrm{B}}$ *respectively.*

Note that forbidden terms of the form $x_{A_iA_{i'}}$ and $x_{B_jB_{j'}}$ do not appear in (4).

As an example, let us see how the I_{3322} inequalities is generated by triangular elimination (see figure 2) of the pentagonal inequality (2). This inequality has two terms $x_{A_1A_2}$ and $x_{B_1B_2}$ which correspond to joint measurements of two observables in one subsystem and are not allowed. Therefore, we eliminate these terms by adding two new nodes $A'_{B_1B_2}$ and $B'_{A_1A_2}$ and adding two triangle inequalities $-x_{A_1A_2} + x_{A_1B'_{A_1A_2}} - x_{A_2B'_{A_1A_2}} \leqslant 0$ and $-x_{B_1B_2} + x_{A'_{B_1B_2}B_1} - x_{A'_{B_1B_2}B_2} \leqslant 0$. If we rewrite the resulting inequality in terms of the vector $\boldsymbol{q}$ instead of the vector $\boldsymbol{x}$ by using isomorphism (1), this inequality becomes the I_{3322} inequality. As we

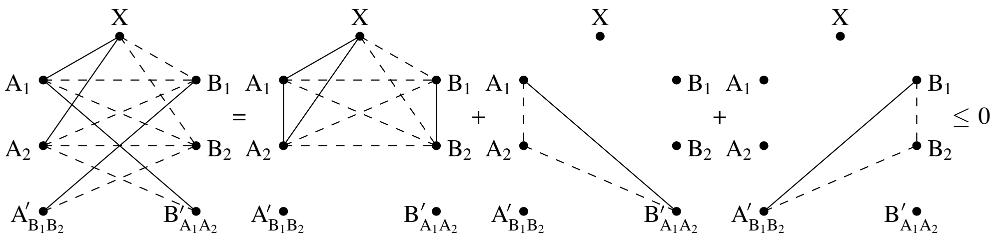

Figure 2. The I_{3322} inequality is generated by triangular elimination from the pentagonal inequality of $\mathrm{CUT}_5^\square$.

will see in the next subsection, this gives another proof of the tightness of the I_{3322} inequality than directly checking the dimension of the face computationally.

2.3. Triangular elimination and facet

In this subsection, we show the main theorem of this paper: under a very mild condition, the triangular elimination of a facet is a facet.

Theorem 2.1. *The triangular elimination of a facet inequality* $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leqslant 0$ *of* $\mathrm{CUT}^\square(K_{1+n_\mathrm{A}+n_\mathrm{B}})$ *is facet inducing for* $\mathrm{CUT}^\square(K_{1,m_\mathrm{A},m_\mathrm{B}})$ *except for the cases that the inequality* $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leqslant 0$ *is a triangle inequality labelled as either* $-x_{XA_1} - x_{XA_2} + x_{A_1A_2} \leqslant 0$ *or* $-x_{A_1A_2} - x_{A_1A_3} + x_{A_2A_3} \leqslant 0$.

For example, as we saw, the CHSH inequality is the triangular elimination of Bell's original inequality, which is a triangle inequality. The I_{3322} inequality, found by Pitowsky and Svozil [11] and Collins and Gisin [12], is the triangular elimination of a pentagonal inequality.

Proof. Let r_F be the set of cut vectors on the hyperplane $\boldsymbol{a}'^{\mathrm{T}}\boldsymbol{x}' = 0$: $r_F = \{\boldsymbol{\delta}'(S') \mid \boldsymbol{a}'^{\mathrm{T}}\boldsymbol{\delta}' = 0, S' : \text{cut}\}$ for $\mathrm{CUT}^\square(K_{1,m_\mathrm{A},m_\mathrm{B}})$. We prove the theorem by exhibiting a linearly independent subset of these cut vectors with cardinality $m_\mathrm{A} + m_\mathrm{B} + m_\mathrm{A}m_\mathrm{B} - 1$.

In the following proof, we consider a simple case of $n_\mathrm{B} = 1$. We consider the other case later. In addition, we assume that $a_{A_iA_{i'}} \leqslant 0$ for all eliminated terms. For the other cases, the proof is similar.

By the above restriction, $m_\mathrm{A} + m_\mathrm{B} + m_\mathrm{A}m_\mathrm{B} - 1 = (n_\mathrm{A}^3 + 3n_\mathrm{A})/2$.

A sketch of proof is as follows: first, we restrict r_F and decompose the whole space of $\mathrm{CUT}^\square(K_{1,m_\mathrm{A},m_\mathrm{B}})$ into two subspaces. For each subspace, we can pick a set of cut vectors which are linearly independent in that subspace. Next, we show that these sets of cut vectors are linearly independent in the whole space.

First, let the subset r'_F of r_F be those cuts such that, for any $1 \leqslant i < i' \leqslant n_\mathrm{A}$, two vertices $A_{i'}$ and $B'_{A_iA_{i'}}$ are assigned the same value. Then, consider the intersection of the space spanned by $\boldsymbol{\delta}'(S') \in r'_F$ and the subspace

$$W = \left\{\left(x_{XA_i}, x_{XB_j}, x_{A_iB_j}, x_{A_iB'_{A_iA_{i'}}}\right)^{\mathrm{T}}_{1\leqslant i<i'\leqslant n_\mathrm{A}, 1\leqslant j\leqslant n_\mathrm{B}}\right\}.$$

From the definition of r'_F, $\delta'_{A_{i'}B'_{A_iA_{i'}}}(S') = 0$. Therefore,

$$\begin{aligned}\boldsymbol{a}'^{\mathrm{T}}\boldsymbol{\delta}'(S') = &\sum_{1\leqslant i\leqslant n_\mathrm{A}} a_{XA_i}\delta_{XA_i}(S') + \sum_{1\leqslant j\leqslant n_\mathrm{B}} a_{XB_j}\delta_{XB_j}(S') \\ &+ \sum_{1\leqslant i\leqslant n_\mathrm{A}, 1\leqslant j\leqslant n_\mathrm{B}} a_{A_iB_j}\delta_{A_iB_j}(S') + \sum_{1\leqslant i<i'\leqslant n_\mathrm{A}} a_{A_iA_{i'}}\delta'_{A_iB'_{A_iA_{i'}}}(S') = 0.\end{aligned}$$

This means that the intersection of space spanned by $\boldsymbol{\delta}'(S') \in r'_F$ and W is equivalent to the space spanned by the cut vectors $r_f = \{\boldsymbol{\delta}(S) \mid \boldsymbol{a}^{\mathrm{T}}\boldsymbol{\delta} = 0, S : \text{cut}\}$ of $\mathrm{CUT}^{\square}(K_{1+n_{\mathrm{A}}+n_{\mathrm{B}}})$. Therefore, from the assumption that the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leqslant 0$ is facet supporting, we can pick $(n_{\mathrm{A}}^2 + 3n_{\mathrm{A}})/2$ linearly independent cut vectors and transform the cut vectors of $\mathrm{CUT}^{\square}(K_{1+n_{\mathrm{A}}+n_{\mathrm{B}}})$ into corresponding cut vectors of $\mathrm{CUT}^{\square}(K_{1,m_{\mathrm{A}},m_{\mathrm{B}}})$. Let this set of linearly independent cut vectors be D_0.

The remaining subspace of $\mathrm{CUT}^{\square}(K_{1,m_{\mathrm{A}},m_{\mathrm{B}}})$ is

$$V = \bigoplus_{i<i'} V_{A_iA_{i'}} = \bigoplus_{i<i'} \left\{ \left(x_{XB'_{A_iA_{i'}}}, x_{A_{i'}B'_{A_iA_{i'}}}, x_{A_{i''}B'_{A_iA_{i'}}} \right)^{\mathrm{T}}_{i''\neq i,i'} \right\}$$

for each eliminated term $A_iA_{i'}$, $1 \leqslant i < i' \leqslant n_{\mathrm{A}}$.

Instead of V, we consider the space

$$V' = \bigoplus_{i<i'} V'_{A_iA_{i'}} = \bigoplus_{i<i'} \left\{ \left(x_{XB'_{A_iA_{i'}}} - x_{A_{i'}B'_{A_iA_{i'}}}, x_{A_{i'}B'_{A_iA_{i'}}}, x_{\alpha_{A_i,A_{i'},A_{i''}}} \right)^{\mathrm{T}}_{i''\neq i,i'} \right\},$$

where

$$x_{\alpha_{A_i,A_{i'},A_{i''}}} = \begin{cases} \frac{1}{2}\left(x_{A_{i''}B'_{A_iA_{i'}}} - x_{A_{i'}B'_{A_{i'}A_{i''}}} - x_{A_{i'}B'_{A_iA_{i'}}} + 3x_{A_{i''}B'_{A_{i'}A_{i''}}} \right) & (i' < i'') \\ \frac{1}{2}\left(x_{A_{i''}B'_{A_iA_{i'}}} - x_{A_{i''}B'_{A_{i''}A_{i'}}} - x_{A_{i'}B'_{A_iA_{i'}}} - x_{A_{i'}B'_{A_{i''}A_{i'}}} \right) & (i'' < i') \end{cases}$$

in the following. Since the transform V to V' is linear, the linear independence of vectors in V is equivalent to that in V'.

Then, we consider the subset $r''_{F,A_iA_{i'}}$ of r_F for each $A_iA_{i'}$ restricted as follows: $A_{i'}$ must be assigned 0 and both $B'_{A_iA_{i'}}$ and A_i must be assigned 1. For other terms $A_{i'''}A_{i''''}(1 \leqslant i''' < i'''' \leqslant n_{\mathrm{A}})$, vertices $A_{i''''}$ and $B'_{A_{i'''}A_{i''''}}$ must be assigned the same value. From that restriction, the equations

$$\begin{aligned} &\delta'_{XB'_{A_iA_{i'}}}(S'') - \delta'_{A_{i'}B'_{A_iA_{i'}}}(S'') = -\delta_{XA_{i'}}(S), \\ &\delta'_{A_{i'}B'_{A_iA_{i'}}}(S'') = 1, \\ &\delta'_{\alpha_{A_i,A_{i'},A_{i''}}}(S'') = -\delta_{A_{i''}A_{i'}}(S) \end{aligned}$$

hold for $\boldsymbol{\delta}'(S'') \in r''_{F,A_iA_{i'}}$. This means that the intersection of the space spanned by $\boldsymbol{\delta}'(S'')$ and the subspace $V'_{A_iA_{i'}}$ is equivalent to that of the space spanned by $\boldsymbol{\delta}(S) \in r_f$ and the subspace

$$U_{A_iA_{i'}} = \left\{ \left(x_{XA_{i'}}, 1, x_{A_{i''}A_{i'}} \right)^{\mathrm{T}}_{i''\neq i,i'} \right\}.$$

Now, because r_f is on the hyperplane $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} = 0$, the above intersection has dimension n_{A} or $n_{\mathrm{A}} - 1$. However, from the condition on the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leqslant 0$, the space spanned by r_f is not parallel to $U_{A_iA_{i'}}$. Therefore, the dimension is n_{A} and we can extract n_{A} cut vectors which are linearly independent in the subspace V' using the cut vectors from r_f. Let this set of cut vectors be $D_{A_iA_{i'}}$.

Finally, we show that $D_0 \cup \bigcup_{1\leqslant i<i'\leqslant n_{\mathrm{A}}} D_{A_iA_{i'}}$ is a linearly independent set of cut vectors. Suppose that the linear combination

$$\sum_{\delta'^{\mathrm{T}}(S')\in D_0} \kappa_{S'}\boldsymbol{\delta}'^{\mathrm{T}}(S') + \sum_{1\leqslant i<i'\leqslant n_{\mathrm{A}}} \sum_{\delta'^{\mathrm{T}}(S'')\in D_{A_iA_{i'}}} \lambda_{S''}^{A_iA_{i'}} \boldsymbol{\delta}'^{\mathrm{T}}(S'') = 0$$

holds. Consider the subspace $V'_{A_iA_{i'}}$ of the above linear combination. From the construction, for D_0 and $D_{A_{i''}A_{i'''}}$, the elements of cut vectors in that subspace are all zero. Therefore, for the linear combination to hold, it must be that $\sum_{\delta'^{\mathrm{T}}(S'')\in D_{A_iA_{i'}}} \lambda_{S''}^{A_iA_{i'}} \boldsymbol{\delta}'^{\mathrm{T}}(S'') = 0$. However, the linear independence of $D_{A_iA_{i'}}$ means that the coefficients are all zero. By repeating this argument, we can conclude that the coefficient $\lambda_{S''}^{A_i A_{i'}}$ must be zero. So, from the linear independence of D_0, the coefficients $\kappa_{S'}$ are also zero. This completes the proof for the case $n_{\mathrm{B}} = 1$.

Now we describe the outline of the proof for general case. The idea of the proof is to perform triangular elimination in two steps: eliminate the edges $A_i A_{i'}$ for $1 \leqslant i < i' \leqslant n_{\mathrm{A}}$ in one step and then the edges $B_j B_{j'}$ for $1 \leqslant j < j' \leqslant n_{\mathrm{B}}$ in the other. To do this, we need the notion of the cut polytope $\mathrm{CUT}^{\square}(G) \subseteq \mathbb{R}^E$ of a general graph $G = (V, E)$, which is obtained from the cut polytope of the complete graph on node set V by removing the coordinates corresponding to the edges missing in E. In particular, we consider the cut polytopes of the following two intermediate graphs: the graph $G_1(n_{\mathrm{A}}, n_{\mathrm{B}})$ obtained from $K_{1,n_{\mathrm{A}},m_{\mathrm{B}}}$ by adding edges $B_j B_{j'}$ and $B_j B'_{A_i A_{i'}}$, and the graph $G_2(n_{\mathrm{A}}, n_{\mathrm{B}})$ obtained from $K_{1,m_{\mathrm{A}},m_{\mathrm{B}}}$ by adding edges $B_j B'_{A_i A_{i'}}$.

The next lemma is a basic fact from polytope theory (see lemma 26.5.2 (ii) in [7]; though the statement there restricts G to be a complete graph, that restriction is not necessary).

Lemma 2.2. *Let G be a graph and G' be a subgraph of G. If $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leqslant 0$ is facet inducing for $\mathrm{CUT}^{\square}(G)$ and $a_e = 0$ for all edges e belonging to G but not to G', then $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leqslant 0$ is facet inducing also for $\mathrm{CUT}^{\square}(G')$.*

The inequality after the first step of triangular elimination is as follows:

$$\sum_{1 \leqslant i \leqslant n_{\mathrm{A}}} a_{XA_i} x_{XA_i} + \sum_{1 \leqslant j \leqslant n_{\mathrm{B}}} a_{XB_j} x_{XB_j} + \sum_{1 \leqslant i \leqslant n_{\mathrm{A}}, 1 \leqslant j \leqslant n_{\mathrm{B}}} a_{A_i B_j} x_{A_i B_j} + \sum_{1 \leqslant i < i' \leqslant n_{\mathrm{A}}} \left(a_{A_i A_{i'}} x_{A_i B'_{A_i A_{i'}}} - \left| a_{A_i A_{i'}} \right| x_{A_{i'} B'_{A_i A_{i'}}} \right) + \sum_{1 \leqslant j < j' \leqslant n_{\mathrm{B}}} a_{B_j B_{j'}} x_{B_j B_{j'}} \leqslant a_0. \tag{5}$$

For the case $n_{\mathrm{B}} = 1$, inequality (5) is exactly the same as (4). We proved above for the case $n_{\mathrm{B}} = 1$ that inequality (5) is facet inducing for $\mathrm{CUT}^{\square}(K_{1,n_{\mathrm{A}},m_{\mathrm{B}}})$. Except for when the original inequality is a triangle inequality, we can extend this argument to prove that inequality (5) is facet inducing also for $\mathrm{CUT}^{\square}(G_1(n_{\mathrm{A}}, n_{\mathrm{B}}))$. This can be generalized for the case $n_{\mathrm{B}} > 1$: inequality (5) is facet inducing for $\mathrm{CUT}^{\square}(G_1(n_{\mathrm{A}}, n_{\mathrm{B}}))$. Then we can repeat a similar argument to prove that the final inequality (4) is facet inducing for $\mathrm{CUT}^{\square}(G_2(n_{\mathrm{A}}, n_{\mathrm{B}}))$. Since $G_2(n_{\mathrm{A}}, n_{\mathrm{B}})$ is a supergraph of the desired graph $K_{1,m_{\mathrm{A}},m_{\mathrm{B}}}$, inequality (4) is facet inducing also for $\mathrm{CUT}^{\square}(K_{1,m_{\mathrm{A}},m_{\mathrm{B}}})$ from lemma 2.2.

2.4. Triangular elimination and symmetry

Many Bell inequalities are equivalent to each other due to the arbitrariness in the labelling of the party, observable and value identifiers. This corresponds to symmetries of the underlying polytope. We consider ways of representing nonequivalent Bell inequalities in this section.

The nonequivalence of Bell inequalities can be translated into two questions about facet inequalities f and f' of a given cut polytope of a complete graph, and their triangular eliminations F and F', respectively:

(i) does the equivalence of f and f' imply the equivalence of F and F'?
(ii) does the equivalence of F and F' imply the equivalence of f and f'?

The answers are both affirmative if we define equivalence appropriately, so equivalence before triangular elimination is logically equivalent to equivalence after triangular elimination. This means that, for example, to enumerate the nonequivalent Bell inequalities, we need only enumerate the facet inequalities of the cut polytope of the complete graph up to symmetry by party, observable and value exchange.

In $\mathrm{CUT}^{\square}(K_{1,m_{\mathrm{A}},m_{\mathrm{B}}})$, the relabelling of all vertices of Alice to that of Bob and vice versa corresponds to a party exchange. On the other hand, the local relabelling of some vertices of Alice (or Bob) corresponds to an observable exchange. Thus, by the observable exchange of Alice represented by the permutation σ over $\{A_1, \ldots, A_{m_{\mathrm{A}}}\}$, an inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leqslant a_0$ is transformed into $\boldsymbol{a}'^{\mathrm{T}}\boldsymbol{x} \leqslant a_0$ where $a'_{\sigma(A_i)V} = a_{A_iV}$ for any vertex V.

In addition, there is an operation which corresponds to a value exchange of some observables, called a *switching* in the theory of cut polytopes. By the switching corresponding to the value exchange of an Alice's observable A_{i_0}, an inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leqslant a_0$ is transformed into $\boldsymbol{a}'^{\mathrm{T}}\boldsymbol{x} \leqslant a_0 - \sum_V a_{A_{i_0}V}$ where $a'_{A_{i_0}V} = -a_{A_{i_0}V}$, and $a'_{A_iV} = a_{A_iV}$ for any $i \neq i_0$ and any vertex $V \neq A_{i_0}$ (definitions for Bob's exchange are similar).

It is well known, and easily shown, that by repeated application of the switching operation we may reduce the right-hand side of any facet inequality to zero.

Let $n_{\mathrm{A}} \leqslant n_{\mathrm{B}}$ and $n = 1 + n_{\mathrm{A}} + n_{\mathrm{B}}$. Let f and f' be facets of $\mathrm{CUT}_n^{\square}$ where the n nodes of K_n is labelled by $V = \{A_1, \ldots, A_{n_{\mathrm{A}}}, B_1, \ldots, B_{n_{\mathrm{B}}}, X\}$. The two facets f and f' are said to be *equivalent* and denoted by $f \sim f'$ if f can be transformed to f' by applying zero or more of the following operations: (1) (only applicable in the case $n_{\mathrm{A}} = n_{\mathrm{B}}$) swapping labels of nodes A_i and B_i for all $1 \leqslant i \leqslant n_{\mathrm{A}}$, (2) relabelling the nodes within $A_1, \ldots, A_{n_{\mathrm{A}}}$, (3) relabelling the nodes within $B_1, \ldots, B_{n_{\mathrm{B}}}$ and (4) switching[4].

Two facets F and F' of $\mathrm{CUT}^{\square}(K_{1,m_{\mathrm{A}},m_{\mathrm{B}}})$ are said to be *equivalent* and denoted by $F \sim F'$ if F can be transformed to F' by applying permutation which fixes node X, switching, or both. This notion of equivalence of facets of $\mathrm{CUT}^{\square}(K_{1,m_{\mathrm{A}},m_{\mathrm{B}}})$ corresponds to equivalence of tight Bell inequalities up to party, observable and value exchange.

Theorem 2.3. *Let the triangular elimination of facet inequalities f and f' be F and F', respectively. Then, $f \sim f' \iff F \sim F'$.*

Proof. A sketch of the proof is as follows. Since the permutation and switching operations are commutative, it is sufficient to prove the proposition under each operation separately. Because the $\Rightarrow$ direction is straightforward for both permutation and switching, we concentrate on the proof of the $\Leftarrow$ direction.

First, consider switching. Suppose F is obtained from a switching of F'. The switching could involve either (i) a new observable introduced by the triangular elimination or (ii) an observable which had a joint measurement term eliminated. Since a switching of type (i) has no effect on f and f', we need only consider type (ii). We can view the triangular elimination of the term $A_iA_{i'}$ as addition of triangle inequality $x_{A_iA_{i'}} - x_{A_iB'_{A_iA_{i'}}} - x_{A_{i}\mathrm{T}B'_{A_iA_{i'}}} \leqslant 0$ or its switching equivalent inequality $-x_{A_iA_{i'}} - x_{A_iB'_{A_iA_{i'}}} + x_{A_{i}\mathrm{T}B'_{A_iA_{i'}}} \leqslant 0$ according to the sign of the coefficient $a_{A_iA_{i'}}$. Thus, if F is switching of F' of vertices A_i and $B'_{A_iA_{i'}}$, then f is switching of f' of A_i.

Next, consider the permutation corresponding to an observable exchange. Observe that for any vertex $A_i (1 \leqslant i \leqslant n_{\mathrm{A}})$, triangular elimination does not change the number of terms A_iV with non-zero coefficient. In addition, it can be shown that for any facet inequality f of the cut polytope of the complete graph other than the triangle inequality, there is no vertex satisfying the following conditions: (a) there are exactly two terms A_iV, with non-zero coefficients, and (b) for those non-zero coefficients a_{A_iW} and a_{A_iU}, $|a_{A_iW}| = |a_{A_iU}|$ [15]. This

[4] The two facets f and f' are said to be *equivalent* and denoted by $f \sim f'$ if f can be transformed to f' by permutation and switching where the permutation τ on V satisfies (1) $\tau(X) = X$ and (2) τ either fixes two sets $\{A_1, \ldots, A_{n_{\mathrm{A}}}\}$ and $\{B_1, \ldots, B_{n_{\mathrm{B}}}\}$ setwise or (in the case $n_{\mathrm{A}} = n_{\mathrm{B}}$) swaps these two sets.

Table 1. The number of inequivalent facets of $\mathrm{CUT}_n^{\square}$ and the number of inequivalent tight Bell inequalities obtained as the triangular eliminations of the facets of $\mathrm{CUT}_n^{\square}$. Asterisk (*) indicates that the value is a lower bound.

n	Facets of $\mathrm{CUT}_n^{\square}$	Tight Bell inequalities via triangular elimination
3	1	2
4	1	2
5	2	8
6	3	22
7	11	323
8	147*	40 399*
9	164 506*	201 374 783*

means that if $F \sim F'$, then the corresponding permutation σ is always in the following form: for permutations τ_A over $\{A_1, \ldots, A_{n_A}\}$ and τ_B over $\{B_1, \ldots, B_{n_B}\}$, $\sigma(A_i) = \tau_A(A_i)$ and $\sigma(B'_{A_i A_{i'}}) = B_{\tau_A(A_i)\tau_A(A_{i'})}$. The situation is the same for Bob.

Therefore, f and f' are equivalent under the permutations τ_A and τ_B. □

2.5. Computational results

By theorem 2.3, we can compute the number of the classes of facets of $\mathrm{CUT}^{\square}(K_{1,m_A,m_B})$ of the same type obtained by applying triangular elimination to non-triangular facets of $\mathrm{CUT}_n^{\square}$. We consulted De Simone, Deza and Laurent [25] for the H-representation of CUT_7, and the 'conjectured complete description' of CUT_8 and the 'description possibly complete' of CUT_9 in SMAPO [20]. The result is summarized in table 1. For $n = 8$ and 9, the number is a lower bound since the known list of the facets of $\mathrm{CUT}_n^{\square}$ is not proved to be complete. A program to generate Bell inequalities from the list in [20] are available from an author's web page at http://www-imai.is.s.u-tokyo.ac.jp/˜tsuyoshi/bell/. The list of the generated Bell inequalities for $n = 8$ is also available.

3. Families of Bell inequalities

While a large list of individual tight Bell inequalities is useful in some applications, a few formulae which give many different Bell inequalities for different values of parameters are easier to treat theoretically. The cut polytope of the complete graph has several classes of valid inequalities whose subclasses of facet-inducing inequalities are partially known (see [7, chapters 27–30] for details). In this section, we apply triangular elimination to two typical examples of such classes to obtain two general formulae for Bell inequalities. In addition, we prove sufficient conditions for these formulae to give a tight Bell inequality.

In this section, terms on the left-hand side of an inequality are arrayed in the format introduced by Collins and Gisin [12]; each row corresponds to coefficients of each observable of party A and each column corresponds to that of party B. Because of switching equivalence, we can assume that the right-hand side of inequality are always zero. The example of the CHSH $-q_{A_1} - q_{B_1} + q_{A_1B_1} + q_{A_1B_2} + q_{A_2B_1} - q_{A_2B_2} \leqslant 0$ is arrayed in the form as follows:

$$\left(\begin{array}{c||cc} & -1 & 0 \\ \hline\hline -1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right) \leqslant 0.$$

3.1. Bell inequalities derived from hypermetric inequalities

Hypermetric inequalities are a fundamental class of inequalities valid for the cut polytope of the complete graph. Here we derive a new family of Bell inequalities by applying triangular elimination to the hypermetric inequalities. A special case of this family, namely, the triangular eliminated pure hypermetric inequality, contains four previously known Bell inequalities: the trivial inequalities like $q_{A_1} \leqslant 1$, the well-known CHSH inequality found by Clauser, Horne, Shimony and Holt [8], the inequality named I_{3322} by Collins and Gisin [12], originally found by Pitowsky and Svozil [11] and the I^2_{3422} inequality by Collins and Gisin [12].

Let s and t be non-negative integers and $b_{A_1}, \ldots, b_{A_s}, b_{B_1}, \ldots, b_{B_t}$ be integers. We define $b_X = 1 - \sum_{i=1}^{s} b_{A_i} - \sum_{j=1}^{t} b_{B_j}$. Then it is known that $\sum_{uv} b_u b_v x_{uv} \leqslant 0$, where the sum is taken over the $\frac{s+t+1}{2}$ edges of the complete graph on nodes $X, A_1, \ldots, A_s, B_1, \ldots, B_t$, is valid for $\mathrm{CUT}^{\square}_{s+t+1}$. This inequality is called the *hypermetric inequality* defined by the weight vector $\boldsymbol{b} = \left(b_X, b_{A_1}, \ldots, b_{A_s}, b_{B_1}, \ldots, b_{B_t}\right)$.

We apply triangular elimination to this hypermetric inequality. Let s_+ and t_+ be the number of positive entries of the forms b_{A_i} and b_{B_j}, respectively. Without loss of generality, we assume that $b_{A_1}, \ldots, b_{A_{s_+}}, b_{B_1}, \ldots, b_{B_{t_+}} > 0$, and $b_{A_{s_+ +1}}, \ldots, b_{A_s}, b_{B_{t_+ +1}}, \ldots, b_{B_t} \leqslant 0$. By assigning $a_{uv} = b_u b_v$ in formula (4), the Bell inequality obtained by triangular elimination is

$$\begin{aligned}
&\sum_{i=1}^{s_+} b_{A_i}\left(\frac{1-b_{A_i}}{2} - \sum_{i'=1}^{i-1} b_{A_{i'}}\right) q_{A_i} + \sum_{i=s_+ +1}^{s} b_{A_i}\left(\frac{1-b_{A_i}}{2} - \sum_{i'=s_+ +1}^{i-1} b_{A_{i'}}\right) q_{A_i} \\
&\quad + \sum_{j=1}^{t_+} b_{B_j}\left(\frac{1-b_{B_j}}{2} - \sum_{j'=1}^{j-1} b_{B_{j'}}\right) q_{B_j} + \sum_{j=t_+ +1}^{t} b_{B_j}\left(\frac{1-b_{B_j}}{2} - \sum_{j'=t_+ +1}^{j-1} b_{B_{j'}}\right) q_{B_j} \\
&\quad + \sum_{j=1}^{t_+} \sum_{j'=t_+ +1}^{t} b_{B_j} b_{B_{j'}} q_{A'_{jj'}} + \sum_{i=1}^{s_+} \sum_{i'=s_+ +1}^{s} b_{A_i} b_{A_{i'}} q_{B'_{ii'}} - \sum_{i=1}^{s} \sum_{j=1}^{t} b_{A_i} b_{B_j} q_{A_i B_j} \\
&\quad - \sum_{1\leqslant i<i'\leqslant s} b_{A_i} b_{A_{i'}} q_{A_i B'_{ii'}} + \sum_{1\leqslant i<i'\leqslant s} \left|b_{A_i} b_{A_{i'}}\right| q_{A_{i'} B'_{ii'}} \\
&\quad - \sum_{1\leqslant j<j'\leqslant t} b_{B_j} b_{B_{j'}} q_{A'_{jj'} B_j} + \sum_{1\leqslant j<j'\leqslant t} \left|b_{B_j} b_{B_{j'}}\right| q_{A'_{jj'} B_{j'}} \leqslant 0.
\end{aligned} \tag{6}$$

Though formula (6) represents a Bell inequality for any choice of weight vector $\boldsymbol{b}$, this Bell inequality is not always tight. Many sufficient conditions for a hypermetric inequality to be facet inducing are known in study of cut polytopes. By theorem 2.1, these sufficient conditions give sufficient conditions for the Bell inequality (6) to be tight. The sufficient conditions stated in [7, corollary 27.2.5] give the following theorem.

Theorem 3.1. *The Bell inequality (6) is tight if one of the following conditions is satisfied.*

(i) For some $l > 1$, the integers $b_{A_1}, \ldots, b_{A_s}, b_{B_1}, \ldots, b_{B_t}$ and b_X contain $l+1$ entries equal to 1 and l entries equal to -1, and the other entries (if any) are equal to 0.

(ii) At least 3 and at most $n-3$ entries in $b_{A_1}, \ldots, b_{A_s}, b_{B_1}, \ldots, b_{B_t}$ and b_X are positive, and all the other entries are equal to -1.

Now we consider some concrete cases when formula (6) represents a tight Bell inequality. If we let $s+t = 2l, s \leqslant l, l > 1, b_{A_1} = \cdots = b_{A_s} = b_{B_1} = \cdots = b_{B_{l-s}} = 1$ and $b_{B_{l-s+1}} = \cdots = b_{B_t} = -1$, then $b_X = 1$ and by case (i) of theorem 3.1, the Bell inequality (6) is tight. In this case, the Bell inequality (6) is in the following form:

$$
\begin{aligned}
&-\sum_{i=1}^{s}(i-1)q_{A_i} - \sum_{j=1}^{l-s}\sum_{j'=l-s+1}^{t} q_{A'_{jj'}} - \sum_{j=1}^{l-s}(j-1)q_{B_j} - \sum_{j=l-s+1}^{t}(j-(l-s))q_{B_j} \\
&\qquad - \sum_{i=1}^{s}\sum_{j=1}^{l-s} q_{A_iB_j} + \sum_{i=1}^{s}\sum_{j=l-s+1}^{t} q_{A_iB_j} - \sum_{1\leqslant i<i'\leqslant s} q_{A_iB'_{ii'}} \\
&\qquad + \sum_{1\leqslant i<i'\leqslant s} q_{A_{i'}B'_{ii'}} - \sum_{1\leqslant j<j'\leqslant l-s} q_{A'_{jj'}B_j} - \sum_{l-s+1\leqslant j<j'\leqslant t} q_{A'_{jj'}B_j} \\
&\qquad + \sum_{j=1}^{l-s}\sum_{j'=l-s+1}^{t} q_{A'_{jj'}B_j} + \sum_{1\leqslant j<j'\leqslant t} q_{A'_{jj'}B_{j'}} \leqslant 0. \qquad (7)
\end{aligned}
$$

Examples of tight Bell inequality in the form (7) are I_{3322} and I_{3422}^2 inequalities [12].

In case of $l = 1$, theorem 3.1 does not guarantee that the Bell inequality (7) is tight. However, in cases of $(l, s, t) = (1, 1, 1)$ and $(1, 1, 2)$, the Bell inequality (7) becomes trivial and CHSH inequalities, respectively, both of which are tight.

Letting $(l, s, t) = (2, 2, 2)$ in (7) gives

$$
-q_{A_2} - q_{B_1} - 2q_{B_2} + q_{A_1B_1} + q_{A_1B_2} + q_{A_2B_1} + q_{A_2B_2} - q_{A_1B'_{12}} + q_{A_2B'_{12}} - q_{A'_{12}B_1} + q_{A'_{12}B_2} \leqslant 0. \qquad (8)
$$

Following the notation in [12], we write inequality (8) by arraying its coefficients:

$$
\left(\begin{array}{cc||ccc}
 & & (A_2) & (A_1) & (A'_{12}) \\
 & & -1 & 0 & 0 \\
\hline\hline
(B_2) & -2 & 1 & 1 & 1 \\
(B_1) & -1 & 1 & 1 & -1 \\
(B'_{12}) & 0 & 1 & -1 & 0
\end{array}\right) \leqslant 0.
$$

Now it is clear that the Bell inequality (8) is the I_{3322} inequality.

Letting $(l, s, t) = (2, 1, 3)$ in (7) gives

$$
\left(\begin{array}{cc||ccc}
 & & (B_2) & (B_3) & (B_1) \\
 & & -1 & -2 & 0 \\
\hline\hline
(A_1) & 0 & 1 & 1 & -1 \\
(A'_{13}) & -1 & 0 & 1 & 1 \\
(A'_{12}) & -1 & 1 & 0 & 1 \\
(A'_{23}) & 0 & -1 & 1 & 0
\end{array}\right) \leqslant 0. \qquad (9)
$$

After exchanging the two values 1 and 0 of the observable A_1, and doing the same to the two values of the observable B_3, the Bell inequality (9) becomes

$$
\left(\begin{array}{cc||ccc}
 & & (B_2) & (\overline{B_3}) & (B_1) \\
 & & 0 & 1 & -1 \\
\hline\hline
(\overline{A_1}) & -1 & -1 & 1 & 1 \\
(A'_{13}) & 0 & 0 & -1 & 1 \\
(A'_{12}) & -1 & 1 & 0 & 1 \\
(A'_{23}) & 1 & -1 & -1 & 0
\end{array}\right) \leqslant 1,
$$

which is the I_{3422}^2 inequality [12]. This means that the Bell inequality (9) is equivalent to the I_{3422}^2 inequality.

3.2. Bell inequalities derived from pure clique-web inequalities

Clique-web inequalities [7, chapter 29] are generalization of hypermetric inequalities. One of the important subclasses of clique-web inequalities is the pure clique-web inequalities, which are always facet inducing. Here we introduce an example of Bell inequalities derived from some pure clique-web inequalities.

For non-negative integers s, t and r with $s \geqslant t \geqslant 2$ and $s - t = 2r$, we consider the pure clique-web inequality with the parameters $n = s + t + 1$, $p = s + 1$, $q = t$ and r. After relabelling the n vertices of K_n by $A_1, \ldots, A_s, X, B_1, \ldots, B_t$ in this order, the Bell inequality (4) corresponding to the clique-web inequality is

$$-\sum_{i=r+1}^{s-r}(i-r-1)q_{A_i} - 2t\sum_{i=s-r+1}^{s} q_{A_i} - \sum_{j=1}^{t}(j-r)q_{B_j} + \sum_{i=1}^{s}\sum_{j=1}^{t} q_{A_iB_j} + \sum_{\substack{1\leqslant i<i'\leqslant s\\ r+1\leqslant j-i\leqslant s-r}}\left(-q_{A_iB'_{ii'}} + q_{A_{i'}B'_{ii'}}\right) + \sum_{1\leqslant j<j'\leqslant t}\left(-q_{A'_{jj'}B_j} + q_{A'_{jj'}B_{j'}}\right) \leqslant 0. \quad (10)$$

The next theorem is a direct consequence of theorem 2.1.

Theorem 3.2. *For any non-negative integers s, t and r with $s \geqslant t \geqslant 2$ and $s - t = 2r$, the Bell inequality (10) is tight.*

3.3. Inclusion relation

Collins and Gisin [12] pointed out that the following I_{3322} inequality becomes the CHSH inequality if we fix two measurements A_3 and B_1 to a deterministic measurement whose result is always 0.

I_{3322}: (the following table) $\leqslant 0$,

		(A_1)	(A_2)	(A_3)
		-1	0	0
(B_1)	-2	1	1	1
(B_2)	-1	1	1	-1
(B_3)	0	1	-1	0

CHSH: (the following table) $\leqslant 0$.

		(A_1)	(A_2)
		-1	0
(B_2)	-1	1	1
(B_3)	0	1	-1

As stated in [12], this fact implies that the CHSH inequality is irrelevant if the I_{3322} inequality is given. In other words, if a quantum state satisfies the I_{3322} inequality with every set of measurements, then it also satisfies the CHSH inequality with every set of measurements.

We generalize this argument and define *inclusion relation* between two Bell inequalities: a Bell inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{q} \leqslant 0$ *includes* another Bell inequality $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{q} \leqslant 0$ if we can obtain the inequality $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{q} \leqslant 0$ by fixing some measurements in the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{q} \leqslant 0$ to deterministic ones.

We do not know whether all the Bell inequalities (except positive probability) include the CHSH inequality. However, we can prove that many Bell inequalities represented by (6) or (10) include the CHSH inequality.

Theorem 3.3. *If $b_{A_1} = b_{A_2} = 1$ and $b_{B_{t+1}} = -1$, then the Bell inequality represented by (6) contains the CHSH inequality.*

Proof. The Bell inequality (6) contains $s + \frac{t}{2}$ observables of Alice and $t + \frac{s}{2}$ observables of Bob. By fixing all but four observables A_1, A_2, B_{t_++1} and B'_{12} to the one whose value is always 0, we obtain the following CHSH inequality: $-q_{A_2} - q_{B_{t_++1}} + q_{A_1B_{t_++1}} + q_{A_2B_{t_++1}} - q_{A_1B'_{12}} + q_{A_2B'_{12}} \leqslant 0$. □

Theorem 3.4. *All the Bell inequalities in the form (10) include the CHSH inequality.*

Proof. By fixing all but four observables A_{r+1}, A_{r+2}, B_{r+1} and $B'_{r+1,r+2}$ to the one whose value is always 0, the Bell inequality (10) becomes the following CHSH inequality: $-q_{A_{r+2}} - q_{B_{r+1}} + q_{A_{r+1}B_{r+1}} + q_{A_{r+2}B_{r+1}} - q_{A_{r+1}B'_{r+1,r+2}} + q_{A_{r+2}B'_{r+1,r+2}} \leqslant 0$. □

3.4. Relationship between I_{mm22} and triangular eliminated Bell inequality

Collins and Gisin [12] proposed a family of tight Bell inequalities obtained by the extension of CHSH and I_{3322} as I_{mm22} family, and conjectured that I_{mm22} is always facet supporting (they also confirmed that for $m \leqslant 7$, I_{mm22} is actually facet supporting by computation). Therefore, whether their I_{mm22} can be obtained by triangular elimination of some facet class of $\mathrm{CUT}^{\square}(K_n)$ is an interesting question.

The I_{mm22} family has the structure as follows:

$$\left(\begin{array}{c||cccccc} & -1 & 0 & \cdots & 0 & 0 & 0 \\ \hline\hline -(m-1) & 1 & 1 & \cdots & 1 & 1 & 1 \\ -(m-2) & 1 & 1 & \cdots & 1 & 1 & -1 \\ -(m-3) & 1 & 1 & \cdots & 1 & -1 & 0 \\ \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ -1 & 1 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 \end{array}\right) \leqslant 0.$$

From its structure, it is straightforward that if I_{mm22} can be obtained by triangular elimination of some facet class of $\mathrm{CUT}^{\square}_n$, then only A_m and B_m are new vertices introduced by triangular elimination, since the other vertices have degree more than 2. For $m = 2, 3, 4$, the I_{mm22} inequality is the triangular elimination of the triangle, pentagon and Grishukhin inequality $\sum_{1\leqslant i<j\leqslant 4} x_{ij} + x_{56} + x_{57} - x_{67} - x_{16} - x_{36} - x_{27} - x_{47} - 2\sum_{1\leqslant i\leqslant 4} x_{i5} \leqslant 0$, respectively. In general, the I_{mm22} inequality is the triangular elimination of a facet-inducing inequality of $\mathrm{CUT}^{\square}_{2m-1}$ and it is tight [26].

3.5. Known tight Bell inequalities other than the triangular elimination of $\mathrm{CUT}^{\square}(K_n)$

Since we have obtained a large number of tight Bell inequalities by triangular elimination of $\mathrm{CUT}^{\square}(K_n)$, the next question is whether they are complete i.e., whether all families and their equivalents form the whole set of facets of $\mathrm{CUT}^{\square}(K_{1,m_{\mathrm{A}},m_{\mathrm{B}}})$.

For the case $m_{\mathrm{A}} = m_{\mathrm{B}} = 3$, the answer is affirmative. Both Śliwa [27] and Collins and Gisin [12] showed that there are only three kinds of inequivalent facets: positive probabilities, CHSH and I_{3322}, corresponding to the triangle facet, the triangular elimination of the triangle facet and the triangular elimination of the pentagonal facet of $\mathrm{CUT}^{\square}(K_n)$, respectively.

On the other hand, in the case $m_{\mathrm{A}} = 3$ and $m_{\mathrm{B}} = 4$, the answer is negative. Collins and Gisin enumerated all of the tight Bell inequalities and classified them into six families of equivalent inequalities [12]. While positive probabilities, CHSH, I_{3322} and I^2_{3422} inequalities are either facets of $\mathrm{CUT}^{\square}(K_n)$ or their triangular eliminations, the other two are not:

$$I^1_{3422} = \begin{pmatrix} & 1 & 1 & -2 \\ \hline 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix} \leqslant 2, \qquad I^3_{3422} = \begin{pmatrix} & 1 & 0 & -1 \\ \hline 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 2 & -1 & -1 & -1 \end{pmatrix} \leqslant 2.$$

4. Concluding remarks

We introduced triangular elimination to derive tight Bell inequalities from the facet inequalities of the cut polytope of the complete graph. Though it does not give the complete list of Bell inequalities, this method derives not only many individual tight Bell inequalities from individual known facet inequalities of cut polytope but also several families of Bell inequalities. This gives a partial answer to the $N = K = 2$ case of the problem posed by Werner [2, problem 1].

Gill poses the following problem in [2, problem 26.B]: is there any Bell inequality that holds for all quantum states, other than the inequalities representing non-negativity of probabilities? Theorems 3.3 and 3.4 give a partial answer to this problem. If a Bell inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{q} \leqslant 0$ includes the CHSH inequality, then the Bell inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{q} \leqslant 0$ is necessarily violated in any quantum states violating the CHSH inequality.

Further investigation of inclusion relation and families of Bell inequalities may be useful to understand the structure of Bell inequalities such as the answer to Gill's problem.

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